BCH modulo [[L,L],[L,L]]

August-27-08
10:41 AM

BCH modulo [[L,L],[L,L]] (cloned on Sep 1, 2008)

Question Find a function F of x and y so that in $\mathcal{U}\left(L/[[L,L],[L,L]]\right)$ $(L:=FL(x,y))$,

$$e^x e^y = \exp(x+y+H(F)) \qquad \left(\begin{array}{l}\text{where } H(f):=H_{[x,y]}(f)\\ :=\mathrm{ad}(f)[x,y]\end{array}\right)$$

Solution Apply $\mathcal{F}$, where $\mathcal{F}(Z):=Z^{-1}EZ$: (the Euler operator)

$$\mathcal{F}(e^x e^y) = e^{-y}e^{-x}E(e^x e^y) = \left(\begin{array}{l}\text{as on}\\ \text{Aug 27}\end{array}\right)$$

$$= x+y+H\left(\frac{1-e^{-y}}{y}\right) = x+y+H(J(y))$$

Aside: $EH(f) = H((2+E)f)$

On the other hand,

$$\mathcal{F}(\exp(x+y+H(F))) =$$

$$= J(\mathrm{ad}(x+y+H(F)))\cdot(x+y+H((2+E)F))$$

$$\left(\text{using } \mathcal{F}(e^g) = \frac{1-e^{-\mathrm{ad}\,g}}{\mathrm{ad}\,g}Eg = J(\mathrm{ad}\,g)Eg\right)$$

$$\left(J(x) = \frac{1-e^{-x}}{x}\right)$$

$$= J(0)(x+y) +$$

$$H\left(\frac{J(x)-J(x+y)}{y} + \frac{J(y)-J(x+y)}{x} - (x+y)FJ'(x+y)\right)$$

$$+H(J(x+y)(2+E)F) =$$

wrong from here (really simpler!)

$$= x+y+H(T_1+T_2-T_3F)$$

$$+H(J(x+y)(2+E)F) = (*)$$

Aside:

$$J(\mathrm{ad}(x+y+H(F)))(H(g)) =$$

$$= H(J(x+y)\,g)$$

$$J(\mathrm{ad}(x+y+H(F)))(x) =$$

$$= J(0)\cdot x +$$

$$+H\left(\frac{J(x)-J(x+y)}{y} - xFJ'(x+y)\right)$$

False

Aside

$$\frac{J(x)-J(x+y)}{y} = \frac{1}{y}\left(\frac{1-e^{-x}}{x} - \frac{1-e^{-(x+y)}}{x+y}\right)$$

$$= \frac{(x+y)(1-e^{-x}) - x(1-e^{-(x+y)})}{} =$$

$$xy(x+y)$$

$$= \frac{y - (x+y)e^{-x} + x e^{-(x+y)}}{xy(x+y)}$$

Likewise, $$\frac{J(y) - J(x+y)}{x} = \frac{x - (x+y)e^{-y} + y e^{-(x+y)}}{xy(x+y)}$$

So $$T_1 + T_2 = \frac{(x+y)(1 - e^{-x} - e^{-y} + e^{-(x+y)})}{xy(x+y)} = \frac{(1-e^{-x})(1-e^{-y})}{xy}$$

$$= J(x)\,J(y)$$

Aside $$J'(x) = \left(\frac{1-e^{-x}}{x}\right)' = \frac{e^{-x}x - (1-e^{-x})}{x^2}$$

$$= \frac{e^{-x}(1+x) - 1}{x^2}$$

So $$T_3 = (x+y)\,J'(x+y) = \frac{e^{-(x+y)}(1+x+y) - 1}{x+y}$$

Aside: solve

$$\left(x\frac{\partial}{\partial x}f + y\frac{\partial}{\partial y}f\right) = L(x,y)f + M(x,y) \qquad (\#)$$

Sol'n: change variables to $x = r\cos\theta$, $y = r\sin\theta$;

get $r\frac{\partial}{\partial r} = r\left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

so $(\#)$ becomes

$$\frac{\partial}{\partial r}f = L(r,\theta)f + M(r,\theta)$$

and this is a linear ODE.